Tensor BM-Decomposition for Compression and Analysis of Spatio-Temporal Third-order Data

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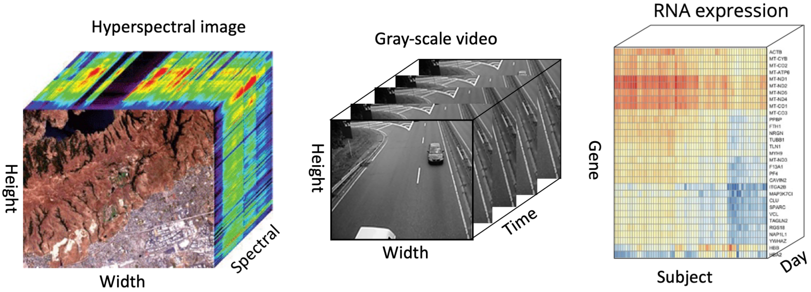
September 30, 2023

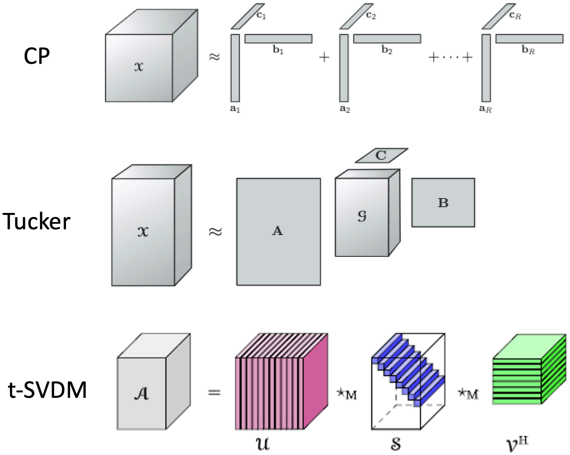
Motivation: Tensors as multi-way arrays

Some real-world data are naturally stored as multi-way data arrays.
Example three-way data



L Ziph Schatzberg: Hyperspectral imaging enables industrial applications, in: Industrial Photonic 2014; Anna Konstorum et al.: Platelet response to influenza vaccination reflects effects of aging, in: Aging Cell 22.2 (2023), e13749.

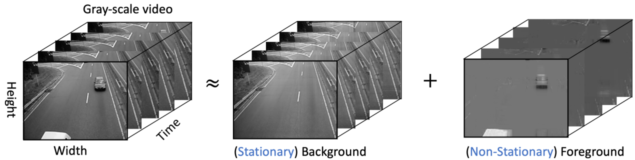
Popular tensor decomposition methods



Misha E Kilmer et al.: Tensor-tensor algebra for optimal representation and compression of multiway data, in: Proceedings of the National Academy of Sciences 118.28 (2021), e2015851118; Tamara G Kolda/Brett W Bader:

Video processing motivated tensor decomposition

Task: decomposing (surveillance) video



Fan Tian et al.: Tensor BM-Decomposition for Compression and Analysis of Spatio-Temporal Third-order Data, in: arXiv preprint arXiv:2306.09201 2023.

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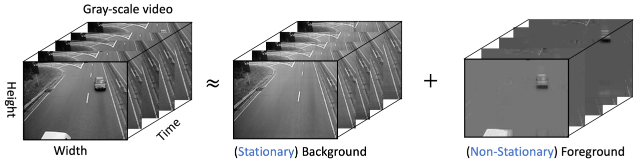

Decomposition based methods:

- background: compressed, well-approximated
- foreground: subtract background from the original video

Tian et al.: Tensor BM-Decomposition for Compression and Analysis of Spatio-Temporal Third-order Data (see n.).

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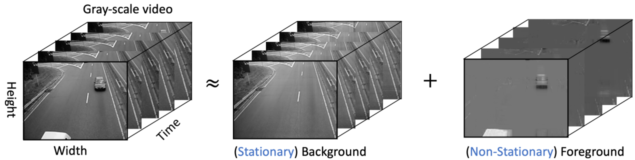

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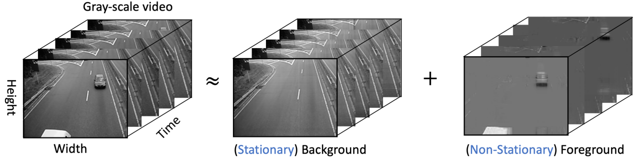

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New tensor method: Bhattacharya-Mesner (BM) decomposition based on tensor BM-product.

Tian et al.: Tensor BM-Decomposition for Compression and Analysis of Spatio-Temporal Third-order Data (see n.).

Bhattacharya-Mesner (BM) Product

Definition. For a third order conformable tensor triplet $\mathbf{A} \in \mathbb{R}^{m\times\ell\times p}$, $\mathbf{B} \in \mathbb{R}^{m\times n\times\ell}$, $\mathbf{C} \in \mathbb{R}^{\ell\times n\times p}$, the BM-product $\mathbf{X} = \mathrm{BMP}(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in \mathbb{R}^{m\times n\times p}$ is given entry-wise by

$$\mathbf{X}[i,j,k] = \sum_{1\le t\le \ell} \mathbf{A}[i,t,k]\,\mathbf{B}[i,j,t]\,\mathbf{C}[t,j,k]$$

Dale M Mesner/Prabir Bhattacharya: Association schemes on triples and a ternary algebra, in: Journal of Combinatorial Theory, Series A 55.2 (1990), pp. 204–234.

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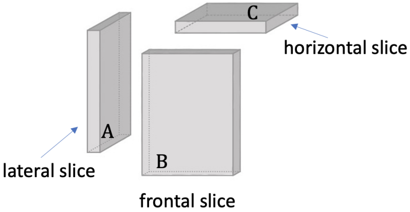
When $\ell = 1$, this describes a BM outer-product of matrix slices.

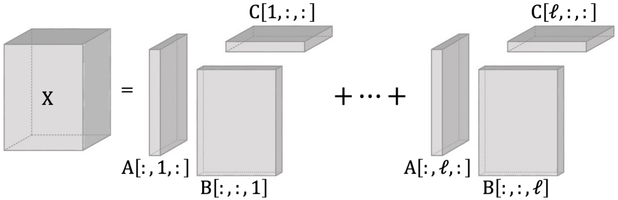


Mesner/Bhattacharya: Association schemes on triples and a ternary algebra (see n.).

Tensor BM-rank

Equivalently, the BM-product can be written as a sum of BM outer-products of matrix slices

$$\mathbf{X} = \sum_{1 \leq t \leq \ell} \mathrm{BMP}\left(\mathbf{A}[:, t, :], \mathbf{B}[:, :, t], \mathbf{C}[t, :, :]\right)$$



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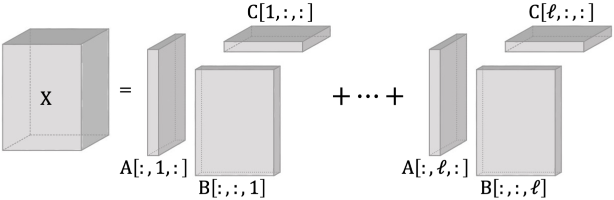

The BM-rank, r, of $\mathbf{X} \in \mathbb{R}^{m \times n \times p}$ is:

- the minimum number of BM outer-products of matrix slices that sum up to $\mathbf{X}$.
- upper bounded by $\min(m, n, p)$.

BM-rank ℓ approximation

Find ℓ, $1 \leq \ell \leq r$, BM-rank 1 terms best approximates $\mathbf{X}$ by solving

$$\min_{\hat{\mathbf{X}}} \|\mathbf{X} - \hat{\mathbf{X}}\|_F^2 \text{ with } \hat{\mathbf{X}} = \sum_{t=1}^{\ell} \mathrm{BMP}\left(\mathbf{A}[:, t, :], \mathbf{B}[:, :, t], \mathbf{C}[t, :, :]\right).$$

where $\|\cdot\|_F$ is the Frobenius norm of $\mathbf{X}$ given by

$$\|\mathbf{X}\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} |\mathbf{X}[i, j, k]|^2}.$$

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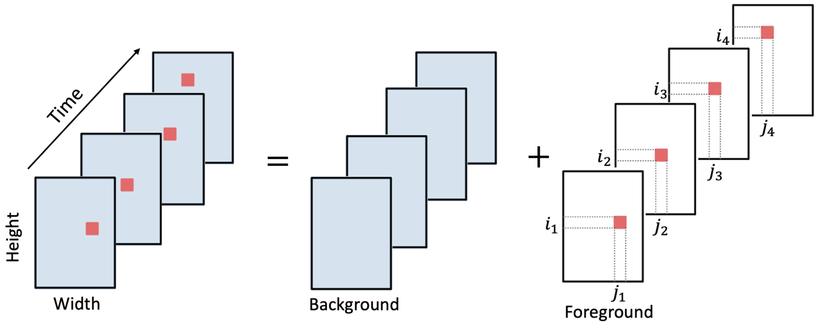
- Solved by Alternating Least-Squares algorithm.
- Achieve good results when provided a good initial guess.

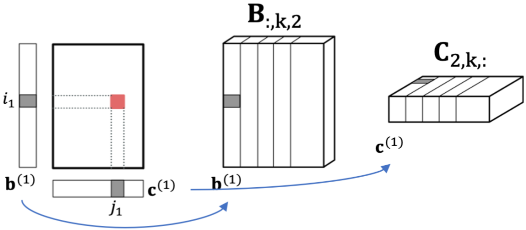
One-pixel spatiotemporal motion model

Assume an object of intensity α and size 1×1 moving on a constant background.
Location at time k is (i_k, j_k).



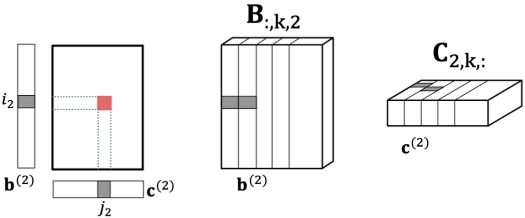
Video foreground motion

Foreground video - frame 1



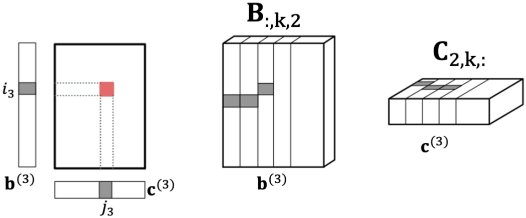
Video foreground motion

Foreground video - frame 2



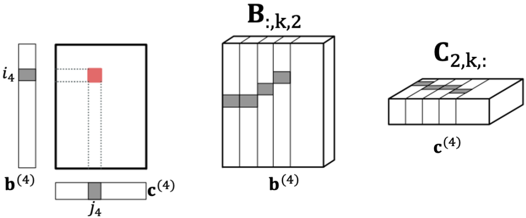
Video foreground motion

Foreground video - frame 3



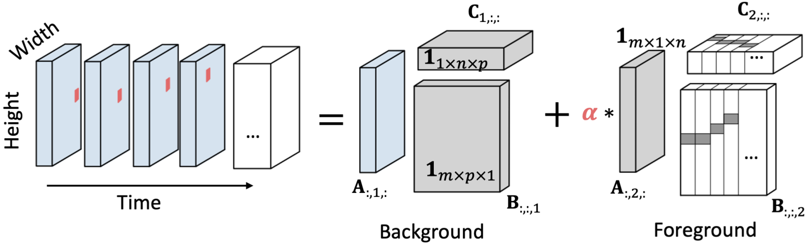
Video foreground motion

Foreground video - frame 4



Tensor reconstruction

Video can be represented exactly by a BM-rank 2 tensor



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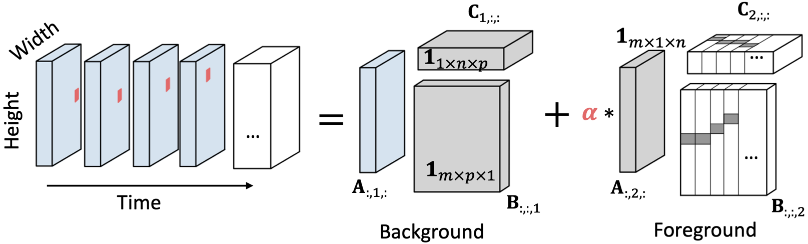

- This model can be generalized with groups of pixels of different intensities all moving.
- Small BM-rank can capture this type of spatiotemporal data.

Application to spatiotemporal third-order data

Real-world video experiment:

Car Video   Escalator Video

Frame 54   Frame 110   Frame 54   Frame 110

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BM-rank ℓ decomposition of video data (order frames as lateral slices):

- Background reconstruction: $\mathbf{X}_{\text{bg}} = \text{BMP}(\mathbf{A}_{:,1,:}, \mathbf{B}_{:,:,1}, \mathbf{C}_{1,:,:})$.
- Foreground reconstruction: $\mathbf{X}_{\text{fg}} = \sum_{t=2}^{\ell} \text{BMP}(\mathbf{A}_{:,t,:}, \mathbf{B}_{:,:,t}, \mathbf{C}_{t,:,:})$.

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Initial guess:

- Spatiotemporal Slice-based SVD (SS-SVD)
- Dynamic Mode Decomposition (DMD)

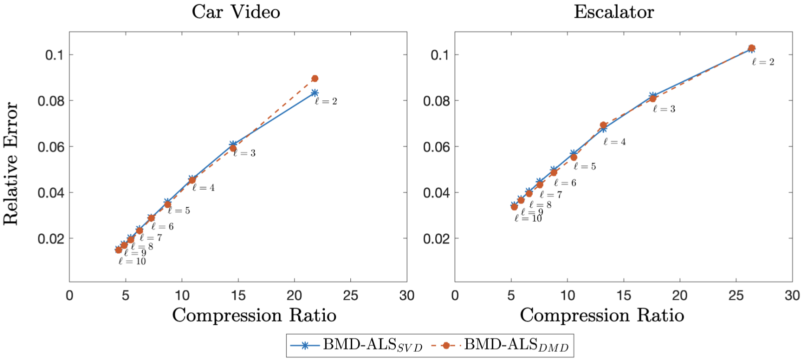
Compression comparison

Car video: $120 \times 120 \times 160$.

Escalator video: $130 \times 200 \times 160$.

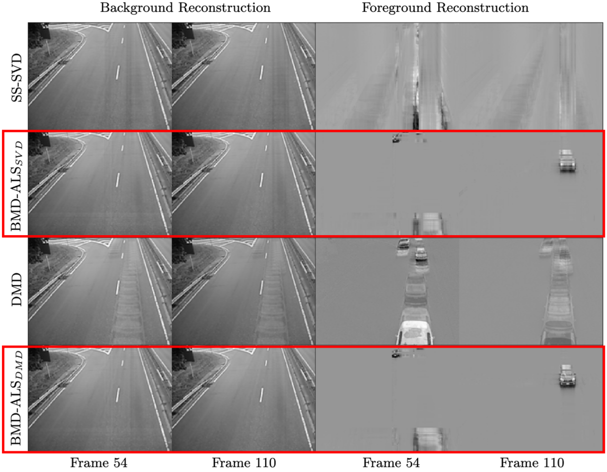
CR$= \frac{\text{uncompressed size}}{\text{compressed size}} = \frac{\ell(mn+mp+np)}{mnp}$; RE$= \frac{\|\mathbf{X}-\hat{\mathbf{X}}\|_F}{\|\mathbf{X}\|_F}$.

Set $\ell = 2, \ldots, 10$.



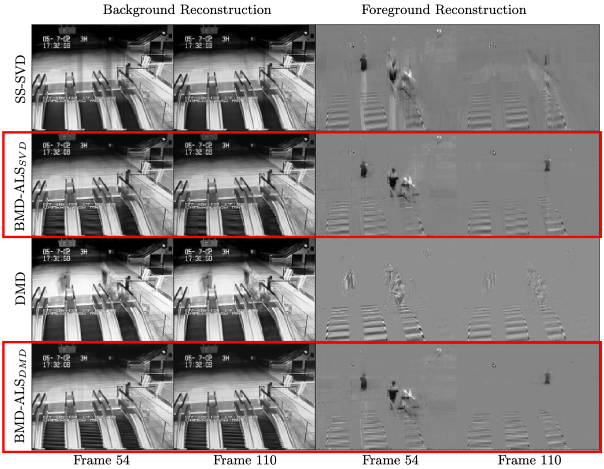
Application to spatiotemporal third-order data

Car video: $120 \times 120 \times 160$.
BM-rank: $\ell = 3$.



Application to spatiotemporal third-order data

Escalator video: $130 \times 200 \times 160$.
BM-rank: $\ell = 5$.



Conclusion and future work

We have

- Introduced the BM-decomposition framework based on tensor BM-product.
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We will

- Apply BM-decomposition to other tasks such as tensor completion.
- Analyse three-way correlations for data with different characteristics in each dimension.

Thank you!